PHYSICAL ANALYSIS TO DETERMINE THE CRITERIA FOR THE DIMENSIONING OF WATER WHEELS

Klaus Brinkmann

University of Hagen, Chair of Electrical Power Engineering
Address for Correspondence:
EniPro, Environmental Process Engineering Dr. Klaus Brinkmann
Leckingser Str. 149, D-58640 Iserlohn, Germany

ABSTRACT

Unfortunately only rudimentary information about the construction and calculation of water wheels from the past are handed down to the posterity. For this reason it makes sense to present a modern kind of physical and mathematical analysis of water wheels. Aim of this paper is to present the fundamental principles for the construction and optimisation of water wheels in a new physical-mathematical way with respect to modern mechanical engineering. This includes all three main types of water wheels as they are under shot, middle shot and over shot water wheels.

INTRODUCTION

An extensive usage of renewable energy sources implies a more decentralised energy supply system structure, because of the more or less stochastic behaviour of their availability. Therefore it is unalterable to combine different energy converter to hybrid systems. These hybrid systems can be single local supply systems but also complex systems with far distance connections of the single different converter units are possible. A further consequent step are interconnections between several hybrid systems to a greater, more abstract kind of complex hybrid system. One possible component for such a hybrid system could be a water wheel. Additionally, a flexible usage of water wheels especially in developing countries should be advantageously, with respect to temporarily changeable local conditions for their installation.

UNTERSHOT WATER WHEEL

Let $\dot{m}\left[kg/s\right]$ be the water flow onto the effective blade area $A\left[m^2\right]$, with a velocity of $v_0\left[m/s\right]$. The resulting force would be $F_0=\rho\cdot A\cdot v_0^2$, with respect to the water density $\rho\left[kg/m^3\right]$. Assuming a circumferential speed of the wheel of $u=\Omega\cdot R\left[m/s\right]$, regarding an angular-speed of $\Omega\left[1/s\right]$ and an effective radius $R\left[m\right]$ of the wheel, the velocity $v_{rel}\left[m/s\right]$ of the water relative to the wheel is given by $v_{rel}=v_0-\Omega\cdot R$. Therefore, the resulting tangential force can be expressed by $F=F_0\cdot\left(1-\Omega\cdot R/v_0\right)^2$.

Together with the angular momentum $M = F \cdot R\,[Nm]$, the formula $P = M \cdot \Omega\,[Nm/s]$ for the power becomes equivalent to the equation $P = F_0 \cdot v_0 \cdot (1-\Omega \cdot R / v_0)^2 \cdot \Omega \cdot R / v_0$. This function gets its maximum at $\Omega \cdot R = v_0 / 3$, with $P_{opt} = 4/27 \cdot \rho \cdot A \cdot v_0^3$. Including the additional hydraulic and mechanical losses η_{hyd} and η_{mech}, the following expression gives the optimum for the reachable power:

$$P_{\max} = \eta_{hyd} \cdot \eta_{mech} \cdot \frac{4}{27} \cdot \rho \cdot A \cdot v_0^3 . \qquad (1)$$

In order to approximate the dynamical behaviour for the start-up period, look at the equilibrium of forces $F_{res} = \frac{M_{res}}{R} = \frac{\Theta_{wheel}}{R} \cdot \frac{d\omega}{d\tau} = F_0 \cdot \left(1 - \frac{\omega \cdot R}{v_0}\right)^2 - F_W$ (*), with $\Theta_{wheel} = \int r^2 dm$ as the moment of inertia, and $F_W = F_R + F_L$, $F_L = M_L / R$ as magnitude of resistance respectively the load.

If the stationary condition $d\omega = 0$ is reached, the resulting force vanishes, this means $F_{res} = 0$. In this case, the angular frequency can be calculated as:

$$\Omega_W = \frac{v_0}{R} \cdot \left[1 - \sqrt{\frac{F_W}{F_0}}\right]. \qquad (2)$$

The rotational speed, which belongs to the maximum power (1), would be reached if the condition $F_W = 4/9 \cdot F_0$ is fulfilled. To work out the start-up function for the angular-speed $\Omega(t)$, the equation of motion (*) must be integrated in the following way:

$$\frac{d\omega}{d\tau} = \frac{F_0 \cdot R}{\Theta_{wheel}} \cdot \left(1 - \frac{\omega \cdot R}{v_0}\right)^2 - \frac{F_W \cdot R}{\Theta_{wheel}} \;\Rightarrow\; \int_0^{\Omega} \frac{d\omega}{\left(1 - \frac{\omega \cdot R}{v_0}\right)^2 - \frac{F_W}{F_0}} = \frac{F_0 \cdot R}{\Theta_{wheel}} \int_0^t d\tau ; \text{ with the substitution}$$

$$\sqrt{\frac{F_W}{F_0}} \le z \equiv \left(1 - \frac{\omega \cdot R}{v_0}\right) \le 1 \;\Rightarrow\; \sqrt{\frac{F_0}{F_W}} \cdot \left[Arcth\left(\sqrt{\frac{F_0}{F_W}}\left(1 - \frac{\Omega \cdot R}{v_0}\right)\right) - Arcth\left(\sqrt{\frac{F_0}{F_W}}\right)\right] = \frac{F_0 \cdot R^2}{v_0 \cdot \Theta_{wheel}} \cdot t \;\Rightarrow$$

$$\Omega(t) = \frac{v_0}{R} \cdot \left\{ 1 - \sqrt{\frac{F_W}{F_0}} \cdot cth\left[\sqrt{\frac{F_W}{F_0}} \cdot \frac{F_0 \cdot R^2}{v_0 \cdot \Theta_{wheel}} \cdot t + Arcth\left(\sqrt{\frac{F_0}{F_W}}\right)\right]\right\}. \qquad (3)$$

One can see, that (3) leads to $\Omega(t = 0) = 0$ and $\Omega(t \to \infty) = \Omega_W$. This means, in this idealistic model, infinite time would be needed to reach the stationary condition. To get a practicable criterion for a more realistic start-up time, look at the reverse function of (3):

$$t = \frac{v_0 \cdot \Theta_{wheel}}{F_0 \cdot R^2} \cdot \sqrt{\frac{F_0}{F_W}} \cdot \left[Arcth\left(\sqrt{\frac{F_0}{F_W}} \cdot \left(1 - \frac{\Omega \cdot R}{v_0}\right)\right) - Arcth\left(\sqrt{\frac{F_0}{F_W}}\right)\right]. \qquad (4)$$

To approximate the start-up time, one can use the condition for the maximum power $P_{\max}$ $\sqrt{F_W / F_0} = 2/3$ and set the frequency to $\Omega = 0{,}9 \cdot v_0 / (3 \cdot R) = 0{,}9 \cdot \Omega_{P\max}$ i.e. 90% of the asymptotic value for $t \to \infty$. This method leads to a final measure for the start-up time, in dependence of hydrodynamic data and constructional parameter:

$$t_{Anlauf}\left(90\%\Omega_{P\max}\right)\cong 1{,}58\cdot\frac{v_0\cdot\Theta_{Rad}}{F_0\cdot R^2}. \quad (5)$$

This formula could be a helpful tool to optimise the constructions (R,Θ) of undershot water wheels with respect to the hydraulic conditions (v_0, F_0).

OVERSHOT WATER WHEEL

If water flows through a rectangular inlet with the dimensions L, $B\,[m]$ and afterwards over the falling height $h\,[m]$ onto the wheel, the mass flow can be given by $\dot{m}=\rho\cdot L\cdot B\cdot\xi\cdot\sqrt{2\cdot g\cdot h}\;[kg/s]$ or $Q=\dot{m}/\rho$ $[m^3/s]$, with ρ as the water density, $g=9{,}81\;[m/s^2]$,and the friction-factor $0<\xi\le 1$. The tangential speed of the wheel is limited by $u\le v_0=\xi\cdot\sqrt{2\cdot g\cdot h}$. With z as the number of cells, each with a of the volume $V_z\;[m^3]$, the rotating wheel has a transport capacity of $\dot{V}=z\cdot V_z/(2\pi)\cdot\Omega$, which is identical with the maximum possible flux $Q_{\max}$. With respect to a partial filling, the real flux can be approximated by $Q=z\cdot\varepsilon\cdot V_z/(2\pi)\cdot\Omega$, with $0\le\varepsilon\le 1$. Then the following relations are valid for stationary conditions:

$$\frac{u}{v_0}=\frac{2\pi\cdot R\cdot L\cdot B}{z\cdot\varepsilon\cdot V_z} \qquad \textit{and} \qquad \Omega=\frac{2\pi\cdot L\cdot B}{z\cdot\varepsilon\cdot V_z}\cdot v_0. \quad (6)$$

During the start-up process, the wheel gets more and more rotations energy with respect to the load momentum and the friction effects. In order to approximate the start-up behaviour, assume, that the wheel would be first filled with water of mass m without rotation, and that the water would be equalised distributed over an angle $\Phi=\varphi_2-\varphi_1$, as shown in figure 1.

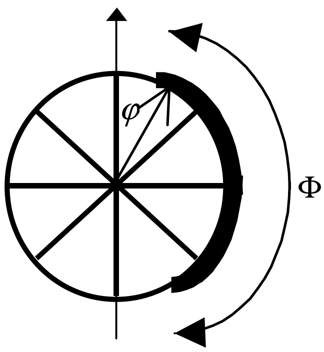

Figure 1: Mass distribution

Equalised distribution $\Rightarrow \quad \frac{m}{\Phi}=\frac{dm}{d\varphi}$

The driving momentum is: $dM = g\cdot dm\cdot R\cdot\sin\varphi$

$$\Rightarrow \quad M=\frac{g\cdot R\cdot m}{\Phi}\int_{\varphi_1}^{\varphi_2}\sin\varphi\cdot d\varphi=m\cdot g\cdot R\cdot\frac{\cos\varphi_1-\cos\varphi_2}{\Phi}.$$

The potential energy is given by:

$$E_P=m\cdot g\cdot R\left(\int_{\varphi_1}^{\varphi_2}\frac{\cos\varphi}{\Phi}\cdot d\varphi-\cos\varphi_2\right).$$

Regarding the additional work $E_R=\mu_R\cdot\left(m_{wheel}+m\right)\cdot g\cdot\varphi\cdot R_S$ against resistances at the shaft with the radius R_S and the relation $\Phi=\int\omega\cdot dt$, the following energy balance can be given:

$$m\cdot g\cdot R\cdot\left(\frac{\sin\varphi_1-\sin\varphi_2}{\Phi}-\cos\varphi_2\right)=\frac{1}{2}\cdot\left(\Theta_{wheel}+m\cdot R^2\right)\cdot\Omega^2+\mu_R\cdot\left(m_{wheel}+m\right)\cdot g\cdot R_S\cdot\varphi. \quad (7)$$

After an initial rotation over the range Φ, the energy conversion produces a starting angular-speed of $\Omega=\Omega_{start}\equiv\sqrt{2\cdot\left(m\cdot g\cdot R\cdot\left(\frac{\sin\varphi_1-\sin\varphi_2}{\Phi}-\cos\varphi_2\right)-\mu_R\cdot m_{total}\cdot g\cdot R_S\cdot\Phi\right)\Big/\left(\Theta_{wheel}+m\cdot R^2\right)}$, which gives the possibility to approximate a start-up time of $t_{start} \approx 2 \cdot \Phi / \Omega_{start}$. Using (7) to make a power balance, the stationary angular-speed can be found as:

$$\Omega = -\frac{\mu_R \cdot m_{ges} \cdot g \cdot R_S}{\dot{m}} + \sqrt{\left(\frac{\mu_R \cdot m_{ges} \cdot g \cdot R_S}{\dot{m} \cdot R^2}\right) + \frac{2 \cdot g}{R} \cdot \left(\frac{\sin\varphi_1 - \sin\varphi_2}{\Phi} - \cos\varphi_2\right)}. \tag{8}$$

The highest possible angular-speed would be realised in the case of $\mu_R = 0$. In combination with the relations (6), these results are valuable for optimisations and constructions of overshot water wheels regarding hydrodynamics as well as mechanical parameter. Neglecting additional hydraulic losses, the following power can be obtained:

$$P_{mech} = M_L \cdot \Omega = \frac{1}{2} \cdot \dot{m} \cdot R^2 \cdot \Omega^2 = \dot{m} \cdot g \cdot R \cdot \left(\frac{\sin\varphi_1 - \sin\varphi_2}{\Phi} - \cos\varphi_2\right) - \mu_R \cdot m_{ges} \cdot g \cdot R_S \cdot \Omega. \tag{9}$$

MIDDLESHOT WATER WHEEL

The working function of a middle shot water wheel can be understand as a combination of the under- and overshot principle. Therefore the here presented model is a physical superposition of two portions of the water flow, with respect to the different action on the wheel, as demonstrated in figure 2. This is equivalent to a separation into a simultaneous potential and dynamical partial-effect.

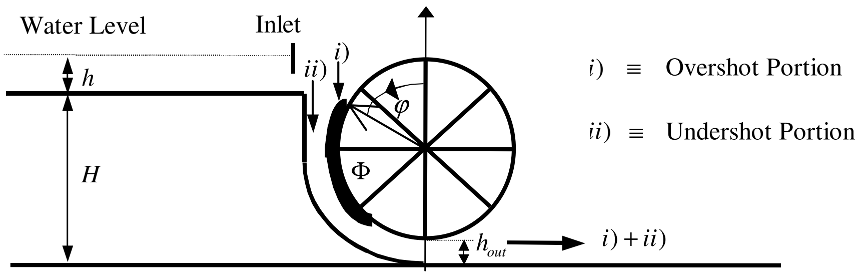

Figure 2: Flow division in two different portions

The total height of the water level is about $H_{tot} \equiv H + h$ $[m]$, so a free flux $v_{out,0}$ gets the speed of $v_{out,0} = \xi_{out} \cdot \sqrt{2 \cdot g \cdot H_{tot}}$ $[m/s]$, with $0 < \xi_{out} \leq 1$. The velocity after the inlet onto the wheel is given by $v_0 = \xi_{In} \cdot \sqrt{2 \cdot g \cdot h}$ $[m/s]$. Analogue to the overshot wheel, the mass flow can be expressed by $\dot{m} = \rho \cdot L \cdot B \cdot \xi_{In} \cdot \sqrt{2 \cdot g \cdot h}$ $[kg/s]$ or $Q = \dot{m}/\rho$ $[m^3/s]$. In order to approximate the start-up behaviour, the non rotating wheel is loaded with the water mass $m_{i)}$ in equalised distribution over the range of Φ. For the two angles φ_1 and φ_2 the conditions $\varphi_1 \approx \arccos\left((H - h_{out})/R - 1\right)$ and $\varphi_2 < 180°$ are obvious. The resulting driving momentum $M_{i)}$ and force $F_{i)}$ is then given by:

$$M_{i)} = \frac{g \cdot R \cdot m_{i)}}{\Phi} \int_{\varphi_1}^{\varphi_2} \sin\varphi \cdot d\varphi = m_{i)} \cdot g \cdot R \cdot \frac{\cos\varphi_1 - \cos\varphi_2}{\Phi} \; [Nm] \Rightarrow F_{i)} = \frac{M_{i)}}{R} \; [N]. \tag{10}$$

If the driving force would only be determined by this gravitational part $i)$, for constructional reasons, the above presented formulas for the overshot water wheel are also valid in this case. Otherwise regard the mass flow $\dot{m}_{ii)}$ $[kg/s]$ onto the effective blade-area $A\left[m^2\right]$ with the velocity $v_{out,0}$ $[m/s]$, resulting in an acting force of $F_0 = \rho \cdot A \cdot v_{out,0}^2$ $[N]$. The circumferential speed of the wheel $u = \Omega \cdot R$ $[m/s]$ leads to a relative velocity to the water flow of $v_{rel} = v_{out,0} - \Omega \cdot R\,[m/s]$. Therefore the dynamical force on the rotating wheel is expressible by $F = F_0 \cdot (1 - \Omega \cdot R / v_{out,0})^2\ [N]$. The equation of motion with both parts *i*) and *ii*) is analogue to (*) given by $F_{tot} = \frac{M_{tot}}{R} = \frac{\Theta}{R} \cdot \frac{d\omega}{d\tau} = F_0 \cdot \left(1 - \frac{\omega \cdot R}{v_0}\right)^2 - F_W + F_{i)}$.

With $F_{eff} = F_R + F_L - F_{i)} = F_W - F_{i)} \Rightarrow \Theta \cdot d\omega / d\tau = F_0 \cdot R \cdot (1 - \omega \cdot R / v_0)^2 - F_{eff} \cdot R$, so that the stationary condition $d\omega = 0$ with $F_{res} = 0$ gives in this case:

$$F_{eff} = F_0 \cdot \left(1 - \frac{\omega \cdot R}{v_{out,0}}\right)^2 \qquad \Rightarrow \qquad \Omega_{stat} = \frac{v_{out,0}}{R} \cdot \left[1 - \sqrt{\frac{F_{eff}}{F_0}}\right]. \tag{11}$$

The formula for the power $P = M \cdot \Omega\,[Nm/s]$ can be expressed with the substitution $x \equiv \Omega \cdot R / v_{out,0}$ as $P = F_W \cdot R \cdot \Omega = F_0 \cdot v_{out,0} \cdot (1 - x)^2 \cdot x + F_{i)} \cdot v_{out,0} \cdot x$. The power is maximal for $x = 2/3 - \sqrt{1/9 - F_{i)} / (3 \cdot F_0)}$. As an essential result of this model is now the possibility to get conditions for the force $F_{i)}$ and therefore for the mass $m_{i)}$ to be loaded:

$$0 \le F_{i)} \le \frac{F_0}{3}, \qquad \text{und} \qquad 0 \le m_{i)} \le \frac{\Phi \cdot \rho \cdot A \cdot v_{out,0}^2}{3 \cdot g \cdot (\cos\varphi_1 - \cos\varphi_2)}. \tag{12}$$

This means for an optimum a three times stronger part of the dynamical portion. In combination with (11) for $F_{eff} = F_W - F_{i)}$ one obtains:

$$\frac{1}{3} \le \sqrt{\frac{F_{eff}}{F_0}} \le \frac{2}{3}, \qquad \text{and} \qquad \frac{F_W}{F_0} = \frac{2}{9} + \frac{2}{3} \cdot \sqrt{\frac{1}{9} - \frac{F_{i)}}{3 \cdot F_0}}. \tag{13}$$

The case $F_{i)} = 0 \Rightarrow x = 1/3 \Rightarrow P_{\max} = \eta_{hyd} \cdot \eta_{mech} \cdot 4/27 \cdot \rho \cdot A \cdot v_{out,0}^3$ in analogy to the undershot water wheel. But if $F_{i)} = F_0 / 3 \Rightarrow x = 2/3$, the maximum power function is given by $P_{\max} = \eta_{hyd} \cdot \eta_{mech} \cdot 8/27 \cdot \rho \cdot A \cdot v_{out,0}^3$, and is surprisingly twice as big as for the pure undershot water wheel. The function for the start-up time can be found in the same way as for the undershot case, as described above:

$$t = \frac{v_{out,0} \cdot \Theta}{F_0 \cdot R^2} \cdot \sqrt{\frac{F_0}{F_{eff}}} \cdot \left[Arcth\left(\sqrt{\frac{F_0}{F_{eff}}} \cdot \left(1 - \frac{\Omega \cdot R}{v_{out,0}}\right)\right) - Arcth\left(\sqrt{\frac{F_0}{F_{eff}}}\right)\right]. \tag{14}$$

In order to approximate a practicable value for the start-up time, one can use the condition for a maximal power $\sqrt{F_{eff} / F_0} = 1/3$ with the frequency $\Omega = 0{,}9 \cdot 2/3 \cdot v_{out,0} / R = 0{,}9 \cdot \Omega_{P\max}$. This leads to an approximate measure for the start-up time, dependant from hydrodynamic and constructional parameter as $t_{start}(90\%\Omega_{P\max}) \cong 2{,}56 \cdot v_{out,0} \cdot \Theta / F_0 \cdot R^2$, and can be used for optimising purposes.

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